On the Asymptotic Behavior of Selfish Transmitters Sharing a Common Channel

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Abstract— This paper analyzes the asymptotic behavior of a multiple-access network comprising a large number of *selfish* transmitters competing for access to a common wireless communication channel, and having different utility functions for determining their strategies. A necessary and sufficient condition is given for the total number of packet arrivals from selfish transmitters to converge in distribution. The asymptotic packet arrival distribution at Nash equilibrium is shown to be a mixture of a Poisson distribution and finitely many Bernoulli distributions.

I. INTRODUCTION

To investigate the behavior of a multiple-access communication network consisting of large number of selfish transmitters, we consider the network model depicted in Fig. 1. In Fig. 1, each transmitter in the transmitter set has an intended receiver in the receiver set. In the context of cellular networks, the transmitter set consists of mobile users requesting uplink reservations to communicate with a base station. In a more general setting, it can be thought of as containing some number of wireless transmitters that are closely located in a wireless ad-hoc network, and that are willing to communicate with another close-by node. The results in this paper can be viewed as characterizing the local behavior of dense wireless networks containing selfish nodes and using a collision channel model at the medium access control (MAC) layer. The collision channel model has been extensively used in the past (e.g., [1], [2]), and it is used to characterize the behavior of networks using no power control and containing nodes with single packet detection capabilities. The protocol model defined in [3] is a variation of the collision model.

A. Game Definition

We assume that transmitter nodes always have packets to transmit, and a transmission fails if there is more than one transmission at the same time. The cost of unsuccessful transmission of node i is $c_i \in (0, \infty)$. A likely meaning that can be attributed to c_i's is the useless power expenditure caused by failed packets. If a transmission is successful, the node that transmitted its packet successfully gets a normalized utility of 1 unit. We model this situation by using a strategic game $G(n, \mathbf{c})$, which is defined formally as follows:
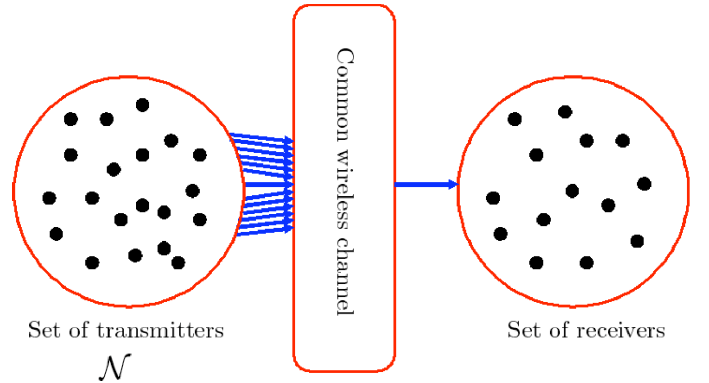
Fig. 1. Network model in which n selfish transmitters contend for the access of a common wireless communication channel to communicate with their intended receivers in the receiver set.

Definition 1: A *heterogenous one-shot random access game* with n transmitter nodes is the game $G(n, \mathbf{c}) = \langle \mathcal{N}, (\mathcal{A}_i)_{i \in \mathcal{N}}, (u_i)_{i \in \mathcal{N}} \rangle$ such that $\mathcal{N} = \{1, 2, \ldots, n\}$ is the set of transmitters, $\mathcal{A}_i = \{0, 1\}$ for all $i \in \mathcal{N}$, where $\mathcal{A}_i$ is the set of actions for node i and 1 means transmission and 0 means back-off, $\mathbf{c} = (c_i)_{i \in \mathcal{N}}$ where c_i is the cost of unsuccessful transmission for node i, and the utility function u_i for all $i \in \mathcal{N}$ is defined as:

$$
\begin{aligned}
u_i(\mathbf{a}) &= 0 && \text{if } a_i = 0, \\
u_i(\mathbf{a}) &= 1 && \text{if } \|\mathbf{a}\|_{l^1} = 1 \text{ and } a_i = 1, \\
u_i(\mathbf{a}) &= -c_i && \text{if } \|\mathbf{a}\|_{l^1} \geq 2 \text{ and } a_i = 1.
\end{aligned}
$$

In Definition 1, $\|\cdot\|_{l^1}$ denotes the l^1 norm for vectors in $\mathbb{R}^n$, and is thus the sum of the absolute values of the components of a vector. If $c_i = c > 0$ for all $i \in \mathcal{N}$, then we will denote $G(n, \mathbf{c})$ by $G(n, c)$, and call it a *homogenous* one-shot random access game.

B. A Note on Notation

$Po(\lambda)$ and $Bern(p)$ will indicate a Poisson distribution with mean λ and a 0-1 Bernoulli distribution with mean p, respectively, as well as the generic random variables with these distributions. For any given two discrete distributions μ and ν on the set of integers $\mathbb{Z}$, $d_V(\mu, \nu)$ denotes the *variational distance* between them, which is defined as $d_V(\mu, \nu) = \sum_{z \in \mathbb{Z}} |\mu(z) - \nu(z)|$. If X and Y are random variables with distributions μ and ν, we sometimes write $d_V(X, Y)$ in stead

of $d_V(\mu, \nu)$ for ease of understanding. If one of the arguments of d_V contains a summation of some random variables, this refers to the convolution of their respective distributions.

If a sequence of probability distributions $\{\mu_n\}_{n=1}^\infty$ converges (in the usual sense of convergence in distribution) to another probability distribution μ, we represent this convergence by $\mu_n \Rightarrow \mu$ as $n \to \infty$.

As in standard terminology, we call a Nash equilibrium a *fully-mixed Nash equilibrium* (FMNE) if all of the transmitters transmit with some positive probability in $(0, 1)$ at this equilibrium. We call a Nash equilibrium a *pure strategy Nash equilibrium* if all transmitters choose their actions deterministically. Therefore, any given transmitter $i \in \mathcal{N}$ either transmits or backs-off with probability one at a pure strategy equilibrium.

$X_i^{(n)}$ is the 0-1 random variable showing the action chosen by transmitter $i \in \mathcal{N}$ when the game $G(n, \mathbf{c})$ is played. As noted before, 0 means back-off, and 1 means transmit. Let $p_{i,n}$ denote the transmission probability of transmitter i when there are n transmitters contending for the channel access. Also let S_n represent the total number of packet arrivals when there are n transmitters contending for the channel access. Note that $S_n = \sum_{i=1}^n X_i^{(n)}$. For a given set $\mathcal{N}_0$, $|\mathcal{N}_0|$ will represent the cardinality of this set.

C. Nash Equilibria of $G(n, \mathbf{c})$

The transmission probability vector at which all nodes back-off with probability one is not a Nash equilibrium of $G(n, \mathbf{c})$ since any node can obtain positive utility by setting its transmission probability to a positive number given the fact that others do not transmit. Therefore, there is an incentive for nodes to deviate from the strategy profile at which all of them back-off. As a result, at Nash equilibria of $G(n, \mathbf{c})$, we expect to observe some of the transmitters transmitting with some positive probabilities and the remaining back-off with probability one. To further investigate this point, let $\pi : \bigcup_{n=2}^\infty \mathbb{R}_+^n \to \mathbb{R}_+$ be such that for any $\mathbf{c} \in \bigcup_{n=2}^\infty \mathbb{R}_+^n$, $\pi(\mathbf{c}) = \prod_i \frac{c_i}{1+c_i}$. The following theorem from [4] characterizes the Nash equilibria of this game.

Theorem 1: Let $X_i^{(n)}$ be the action chosen by transmitter $i \in \mathcal{N}$, $\mathbf{c} \in \mathbb{R}^n$ and $\mathcal{N}_0 \subseteq \mathcal{N}$ with $2 \leq |\mathcal{N}_0| \leq n$. Then, $G(n, \mathbf{c})$ has n pure-strategy Nash equilibria. Moreover, any mixed-strategy profile such that nodes in $\mathcal{N}_0$ transmit with some positive probability, and nodes in $\mathcal{N} - \mathcal{N}_0$ back-off with probability 1 is a Nash equilibrium if and only if $\mathbb{P}\{X_i^{(n)} = 1\} = 1 - (\frac{1+c_i}{c_i})(\pi(\mathbf{c}'))^{\frac{1}{|\mathcal{N}_0|-1}}$ for $i \in \mathcal{N}_0$, and $\frac{c_i}{1+c_i} \overset{(\geq)}{>} \pi(\mathbf{c}')^{\frac{1}{|\mathcal{N}_0|-1}}$ for all $i \in \mathcal{N}$ (with $\geq$ if $i \in \mathcal{N} - \mathcal{N}_0$), where $\mathbf{c}' = (c_i)_{i \in \mathcal{N}_0}$.

D. Review: Homogenous Case

We briefly mention the form of the asymptotic distribution of the total number of packet arrivals when all transmitters have identical utility functions. In this case, the necessary and sufficient condition given in Theorem 1 can be satisfied for any subset $\mathcal{N}_0$ of $\mathcal{N}$ with $|\mathcal{N}_0| \geq 2$ for proper choice of the nodes' transmission probabilities. Therefore, for any given $\mathcal{N}_0 \subseteq \mathcal{N}$ with $|\mathcal{N}_0| \geq 2$, a mixed strategy Nash equilibrium at which only the transmitters in $\mathcal{N}_0$ transmit with some positive probability, and the rest of them back-off with probability one exists. At such a Nash equilibrium, the transmission probabilities of transmitters in $\mathcal{N}_0$ are all equal to $p = 1 - \left(\frac{c}{1+c}\right)^{\frac{1}{|\mathcal{N}_0|-1}}$. Thus, transmitters transmit with probability $p = 1 - \left(\frac{c}{1+c}\right)^{\frac{1}{n-1}}$ at the FMNE. Hence, at the FMNE of the homogenous random access game, S_n becomes a binomial random variable with the success probability $p = 1 - \left(\frac{c}{1+c}\right)^{\frac{1}{n-1}}$. Since $n \cdot \left(1 - \left(\frac{c}{1+c}\right)^{\frac{1}{n-1}}\right)$ approaches to $-\log\left(\frac{c}{1+c}\right)$ as n goes to infinity, S_n converges, in distribution, to a Poisson distribution with mean $-\log\left(\frac{c}{1+c}\right)$, which can be shown by using Poisson approximation the binomial distribution ([8]).

Further details can be found in [4]. For the rest of the paper, our aim is to prove the limit theorem for S_n in the more general case when nodes do not have identical utility functions. We first give a counter example showing that the limiting distribution of S_n cannot always be a pure Poisson distribution. In this latter case, we then, however, show that it can be arbitrarily closely approximated in distribution by a summation of finitely many independent Bernoulli random variables and a Poisson random variable.

E. Related Work

Two closely related work are [5] and [6]. In these work, they analyze the performance of Slotted ALOHA protocol with selfish transmitters by only considering the homogenous case where selfish nodes have identical utility functions. Moreover, they do not provide any results regarding the asymptotic packet arrival distribution. In [4], we mostly focused on the asymptotic channel throughput and the asymptotic packet arrival distribution in the homogenous case for the same problem set-up. We also provided a weaker necessary condition for the convergence of packet arrivals in distribution in the heterogeneous case. Different from the existing work in the literature, this paper will concentrate on the asymptotic packet arrival distribution in the more general case when selfish transmitters having different utility functions contend for the access of a common wireless communication channel. We provide a necessary and sufficient condition for the convergence of total number of packet arrivals in distribution as the number of selfish transmitters increases to infinity. We also specify the form of the asymptotic packet arrival distribution.

II. LIMITING BEHAVIOR OF S_n IN THE HETEROGENOUS CASE

We start our discussion with an example illustrating that the Poisson type convergence does not occur in general in the heterogeneous case. This result, while somewhat negative, will shed light on the form of the limiting distributions for S_n. In this example, the limiting distribution of the packet arrivals will be a mixture of a Poisson distribution and several Bernoulli distributions.

Example: We consider the FMNE of the one-shot random access game, and let $\mathbf{c}_n = (M_1, M_2, \ldots, M_l, \underbrace{1, 1, \ldots, 1}_{n-l \text{ of them}})$. By

Theorem 1, $G(n, \mathbf{c}_n)$ has an FMNE if and only if the following conditions are satisfied:

$$\frac{M_i}{1+M_i} > \left(\frac{1}{2}\right)^{\frac{n-l}{n-1}} \prod_{j=1}^{l} \left(\frac{M_j}{1+M_j}\right)^{\frac{1}{n-1}} \quad \text{for } 1 \le i \le l, \,(1)$$

and

$$\left(\frac{1}{2}\right)^{l-1} > \prod_{j=1}^{l} \frac{M_j}{1+M_j} \quad \text{for } l+1 \le i \le n. \tag{2}$$

Since the right-hand side of (1) approaches to $\frac{1}{2}$, we must choose $M_i > 1$ for all $i \in \{1, 2, ..., l\}$ to have the FMNE for all n large enough. Any choice of $M_1, M_2, \ldots, M_l$ such that $M_i > 1$ for all $i \in \{1, 2, ..., l\}$ and $\prod_{j=1}^{l} \frac{M_j}{1+M_j} < (\frac{1}{2})^{l-1}$ is good for our purposes. One way of choosing such M_i's is to make all of the $\frac{M_i}{1+M_i}$'s smaller than $(\frac{1}{2})^{\frac{l-1}{l}}$, which corresponds to $M_1, M_2, \ldots, M_l \in \left(1, \frac{1}{2^{\frac{l-1}{l}}-1}\right)$.

For appropriately chosen M_i, $1 \le i \le l$, we have the following transmission probabilities:

$$p_{i,n} = 1 - \frac{1+M_i}{M_i}\left(\frac{1}{2}\right)^{\frac{n-l}{n-1}} \prod_{j=1}^{l}\left(\frac{M_j}{1+M_j}\right)^{\frac{1}{n-1}} \quad \text{for } 1 \le i \le l,$$

and

$$p_{i,n} = 1 - 2^{\frac{l-1}{n-1}} \prod_{j=1}^{l}\left(\frac{M_j}{1+M_j}\right)^{\frac{1}{n-1}} \quad \text{for } l+1 \le i \le n.$$

Define $Y_n = \sum_{i=l+1}^{n} X_i^{(n)}$. Then, $S_n = \sum_{i=1}^{l} X_i^{(n)} + Y_n$. Observe that $p_{\max}^{(n)} \stackrel{\triangle}{=} \max_{l+1 \le i \le n} p_{i,n} \to 0$ and

$$\sum_{i=l+1}^{n} p_{i,n} \to \log(2^{1-l}) + \sum_{j=1}^{l} \log\left(1 + \frac{1}{M_j}\right) \tag{3}$$

as $n \to \infty$. Therefore,

$$Y_n \Rightarrow Po\left(\log(2^{1-l}) + \sum_{j=1}^{l} \log\left(1 + \frac{1}{M_j}\right)\right), \tag{4}$$

$$X_i^{(n)} \Rightarrow Bern\left(1 - \frac{1+M_i}{2M_i}\right) \quad \text{for } 1 \le i \le l. \tag{5}$$

As a result, we conclude, by using the continuity theorem and the independence of the random variables Y_n and $X_i^{(n)}$, that

$$S_n \Rightarrow Po\left(\log(2^{1-l}) + \sum_{i=1}^{l} \log\left(1 + \frac{1}{M_i}\right)\right)$$
$$+ \sum_{i=1}^{l} Bern\left(1 - \frac{1+M_i}{2M_i}\right). \tag{6}$$

One interesting feature of this example is that we cannot find infinitely many M_i's that are uniformly bounded away from 1, since $\frac{1}{2^{\frac{l-1}{l}}-1} \to 1$ as $l \to \infty$. This observation will help us in obtaining the asymptotic distribution of S_n in the heterogeneous case.

For the rest of the paper, we focus on the asymptotic distribution of S_n at the FMNE of $G(n, \mathbf{c}_n)$ since the FMNE is the fairest Nash equilibrium at which all transmitters have a chance to transmit with some positive probability depending on their costs of failed transmissions. More general results can
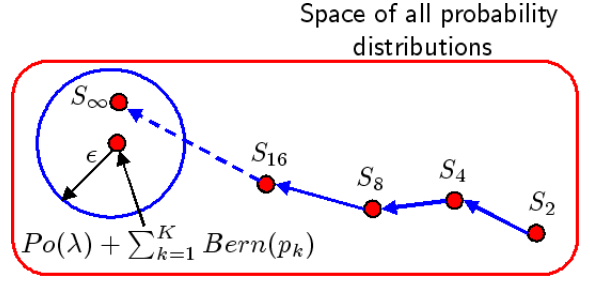

Fig. 2. A pictorial explanation of Theorem 2. The limiting distribution of S_n lies in a small ball around the distribution of the random variable $Po(\lambda) + \sum_{k=1}^{K} Bern(p_k)$.

be found in [7]. Set $a_i = \frac{c_i}{1+c_i}$ and $a_{\min}(n) = \min_{1 \le i \le n} a_i$. We will assume that the costs of unsuccessful transmission of the nodes depend only on their internal parameters such as remaining battery lifetime or energy spent per transmission. Therefore, adding new transmitters to the game does not change the costs of the transmitters already playing the game. Thus, $\alpha = \inf_{i \ge 1} a_i = \lim_{n \to \infty} a_{\min}(n)$ is well-defined. The following two auxiliary results will help in proving the main theorem, Theorem 2, of the paper. Their proofs can be found in [4] and [7]. The first one states the convergence of the geometric mean of the numbers $a_1, a_2, \ldots, a_n$ to a constant $\alpha > 0$ as $n \to \infty$ if the FMNE exists for all $n \ge 2$. The second one states the convergence of the a_i's to the same constant α if the FMNE exists for all $n \ge 2$.

Lemma 1: Let $Geo(a_1, a_2, \ldots, a_n)$ denote the geometric mean of $a_1, a_2, \ldots, a_n$. If the FMNE exists for all $n \ge 2$ exists, then $\lim_{n \to \infty} Geo(a_1, \ldots, a_n) = \alpha > 0$.

Lemma 2: If the FMNE exists for all $n \ge 2$, then $\lim_{i \to \infty} a_i = \alpha$.

In words, Lemma 2 says that if the FMNE exists for all $n \ge 2$, then we can find a $c > 0$ such that for any given $\delta > 0$, the costs of all the transmitters, except for the finitely many of them, incurred as a result of unsuccessful transmissions are concentrated in $(c - \delta, c + \delta)$. Intuitively, we anticipate the selfish nodes whose costs lie in $(c - \delta, c + \delta)$ to behave as in the homogeneous case. Thus, the total number of packet arrivals from these nodes can be approximated by a Poisson random variable up to an arbitrarily small error term $\epsilon(\delta)$ depending on δ. The arrivals from the other finitely many nodes whose costs lie outside of $(c - \delta, c + \delta)$ can be given by a summation of finitely many Bernoulli random variables. Therefore, we expect that once S_n converges in distribution, for any given $\epsilon > 0$, we should be able to find a Poisson random variable $Po(\lambda)$ and finitely many Bernoulli random variables $\{Bern(p_j)\}_{j=1}^{K}$ such that S_n can be approximated, in variational distance, by the sum of $Po(\lambda)$ and $\{Bern(p_j)\}_{j=1}^{K}$ up to an error term less than ϵ. A pictorial representation of this fact is given in Fig. 2.

The main result of the paper formally stating the above observation is given in Theorem 2. In the proof of Theorem 2, we let $p_{i,\infty} = \lim_{n \to \infty} p_{i,n}$ when the FMNE exists for all $n \ge 2$. Existence of this limit can be shown by using Lemma 1.

Theorem 2: Assume FMNE exists for all $n \ge 2$. Then, S_n

converges in distribution if and only if $\lim_{n\to\infty}\sum_{i=1}^{n}p_{i,n}\in(0,\infty)$. *Moreover, for any $\epsilon > 0$, there exists a Poisson random variable $Po(\lambda)$ and a collection of finitely many Bernoulli random variables $\{Bern(p_k)\}_{k=1}^{K}$ such that*

$$\limsup_{n\to\infty} d_V\left(S_n, Po(\lambda) + \sum_{k=1}^{K} Bern(p_k)\right) \le \epsilon. \qquad (7)$$

Proof: $\Longleftarrow$: We first show the *if* direction. Suppose $\lim_{n\to\infty}\sum_{i=1}^{n}p_{i,n} = m \in (0,\infty)$ exists. Let $m_n = \sum_{i=1}^{n}p_{i,n}$ and S_n be distributed according to μ_n. We will first show that $\{\mu_n\}_{n=1}^{\infty}$ is a tight sequence of distributions. To this end, we show that for each $\epsilon > 0$, $\exists M \in \mathbb{N}$ such that $\mathbb{P}\{S_n \in [0, M]\} \ge 1 - \epsilon$. Choose a $\delta > 0$ and choose $N \in \mathbb{N}$ large enough that $m_n \in [m - \delta, m + \delta]$ for all $n \ge N$. Then, by the Markov inequality,

$$\mathbb{P}\{S_n > M\} \le \frac{\mathbb{E}\left[(S_n - m_n)^2\right]}{(M - m_n)^2}.$$

We bound $\mathbb{E}\left[(S_n - m_n)^2\right]$ as follows:

$$\mathbb{E}\left[(S_n - m_n)^2\right] = \sum_{i=1}^{n} Var\left(X_i^{(n)}\right) \le m_n \le m + \delta.$$

In addition, $(M - m_n)^2 \ge (M - m - \delta)^2$. Thus,

$$\mathbb{P}\{S_n > M\} \le \frac{m + \delta}{(M - m - \delta)^2}.$$

If M is large enough, then we have $\mathbb{P}\{S_n > M\} \le \epsilon$ for all $n \ge N$. By making M larger, if necessary, we have $\mathbb{P}\{S_n > M\} = 0$ for all $n < N$. As a result, $\mathbb{P}\{S_n \in [0, M]\} > 1 - \epsilon$ for all n. Thus, $\{\mu_n\}_{n=1}^{\infty}$ is a tight sequence of distributions. Now, we will show that μ_n converges, in variational distance, to a distribution μ. This fact, combined with tightness of $\{\mu_n\}_{n=1}^{\infty}$, will imply that μ is in fact a probability distribution and $\mu_n \Rightarrow \mu$. By using Lemma 1 and Lemma 2, it can be shown that $\lim_{i\to\infty} p_{i,\infty} = 0$. Thus, for any given $\epsilon > 0$, we can choose K large enough that $\max_{i\ge K} p_{i,\infty} \le \frac{\epsilon}{8m}$.

Let $\lambda_n = \sum_{i=K}^{n} p_{i,n}$ and $\lambda = \lim_{n\to\infty}\lambda_n$. Then, by using the properties of variational distance, $d_V\left(S_n, Po(\lambda) + \sum_{i=1}^{K-1} Bern(p_{i,\infty})\right)$ can be bounded above as

$$d_V\left(S_n, Po(\lambda) + \sum_{i=1}^{K-1} Bern(p_{i,\infty})\right)$$
$$\le d_V\left(\sum_{i=1}^{K-1} X_i^{(n)}, \sum_{i=1}^{K-1} Bern(p_{i,\infty})\right)$$
$$\quad + d_V\left(Po(\lambda_n), Po(\lambda)\right) + 2\sum_{i=K}^{n} p_{i,n}^2$$
$$\le d_V\left(\sum_{i=1}^{K-1} X_i^{(n)}, \sum_{i=1}^{K-1} Bern(p_{i,\infty})\right)$$
$$\quad + d_V\left(Po(\lambda_n), Po(\lambda)\right) + 2\max_{K\le i\le n} p_{i,n}\sum_{i=K}^{n} p_{i,n}.$$

Thus,

$$\limsup_{n\to\infty} d_V\left(S_n, \sum_{i=1}^{K-1} Bern(p_{i,\infty}) + Po(\lambda)\right)$$
$$\le 2\limsup_{n\to\infty}\left(\lambda_n.\max_{K\le i\le n} p_{i,n}\right)$$
$$= 2\lambda\limsup_{n\to\infty}\max_{K\le i\le n} p_{i,n} \quad (\text{since } \lambda_n \to \lambda).$$

Let $i(n)$ be such that $p_{i(n),n} = \max_{K\le i\le n} p_{i,n}$. Then, there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} p_{i(n_k),n_k} = \limsup_{n\to\infty}\max_{K\le i\le n} p_{i,n}.$$

If $\{i(n_k)\}_{k=1}^{\infty}$ is a bounded sequence, there exists a further subsequence $\{i(n_{k_j})\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} i(n_{k_j}) = i^{**}$. Since we are considering a sequence of integers converging to another integer, there exists $N \in \mathbb{N}$ such that we have $i(n_{k_j}) = i^{**}$ for all $j \ge N$. Thus,

$$\limsup_{n\to\infty}\max_{K\le i\le n} p_{i,n} = p_{i^{**},\infty} \le \max_{i\ge K} p_{i,\infty}.$$

If $\{i(n_k)\}_{k=1}^{\infty}$ is not a bounded sequence, then there exists a further subsequence $\{i(n_{k_j})\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} i(n_{k_j}) = \infty$. Let $\gamma_n = (\prod_{i=1}^{n} a_i)^{\frac{1}{n-1}}$. Observe that transmission probabilities at FMNE can be given as $p_{i,n} = 1 - a_i^{-1}\gamma_n$. So,

$$\limsup_{n\to\infty}\max_{K\le i\le n} p_{i,n} = \lim_{j\to\infty} p_{i(n_{k_j}),n_{k_j}}$$
$$= 1 - \lim_{j\to\infty} a_{i(n_{k_j})}^{-1}\lim_{j\to\infty}\gamma_{n_{k_j}}$$
$$= 0 \le \max_{i\ge K} p_{i,\infty}.$$

Therefore,

$$\limsup_{n\to\infty} d_V\left(S_n, Po(\lambda) + \sum_{i=1}^{K-1} Bern(p_{i,\infty})\right)$$
$$\le 2\lambda\max_{i\ge K} p_{i,\infty} \le \frac{\epsilon}{4}.$$

Thus, $\exists N \in \mathbb{N}$ large enough so that

$$d_V\left(S_n, Po(\lambda) + \sum_{i=1}^{K-1} Bern(p_{i,\infty})\right) \le \frac{\epsilon}{2}$$

for all $n \ge N$. As a result, we conclude that $\{\mu_n\}_{n=1}^{\infty}$ is a Cauchy sequence with respect to the metric d_V on the set of all probability measures $\mathcal{Z}$ on $\mathbb{Z}$. This also implies that $\{\mu_n(z)\}_{n=1}^{\infty}$ is a Cauchy sequence for all $z \in \mathbb{Z}$, and therefore, converges for any $z \in \mathbb{Z}$. Let $\mu(z) = \lim_{n\to\infty}\mu_n(z)$ for all $z \in \mathbb{Z}$. This combined with the tightness of $\{\mu_n\}_{n=1}^{\infty}$ implies that μ is a probability measure and $\mu_n \Rightarrow \mu$.

$\Longrightarrow$: Now, we prove the *only if* part. In fact, this will be a general result for any sequence of triangular arrays of Bernoulli random variables. Suppose now that there exists an $\mathbb{R}$ valued random variable S_∞ such that S_n converges in distribution to S_∞. First, assume

$$\limsup_{n\to\infty} m_n = \infty,$$

and let $Y_i^{(n)} = X_i^{(n)} - p_{i,n}$. Set $R_n = \sum_{i=1}^{n} Y_i^{(n)}$. Consider $\mathbb{E}\left[e^{-tY_i^{(n)}}\right]$ for $t > 0$. We have

$$\mathbb{E}\left[e^{-tY_i^{(n)}}\right]$$
$$\leq 1 + \frac{1}{2!}t^2 \left|\mathbb{E}\left[(Y_i^{(n)})^2\right]\right| + \frac{1}{3!}t^3 \left|\mathbb{E}\left[(Y_i^{(n)})^3\right]\right| + \cdots.$$

We will show $\left|\mathbb{E}[(Y_i^{(n)})^k]\right| \leq p_{i,n}$ for all k. For $k = 2$,

$$\mathbb{E}\left[(Y_i^{(n)})^2\right] = Var\left(X_i^{(n)}\right) \leq \mathbb{E}\left[\left(X_i^{(n)}\right)^2\right] = p_{i,n}.$$

For any $k \geq 3$, we have

$$\left|\mathbb{E}\left[\left(Y_i^{(n)}\right)^k\right]\right|$$
$$\leq \left(\mathbb{E}\left[\left|Y_i^{(n)}\right|^{2k-2}\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\left|Y_i^{(n)}\right|^2\right]\right)^{\frac{1}{2}} \quad \text{(Hölder's Ineq.)}$$
$$\leq \left(\mathbb{E}\left[\left|Y_i^{(n)}\right|^2\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\left|Y_i^{(n)}\right|^2\right]\right)^{\frac{1}{2}} \leq p_{i,n}.$$

Thus,

$$\mathbb{E}\left[e^{-tY_i^{(n)}}\right] \leq 1 + \frac{1}{2!}t^2 p_{i,n} + p_{i,n}\left(\frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right). \quad (8)$$

Then, there is a $\delta_1 > 0$ such that, uniformly over all $p_{i,n}$, we have

$$\mathbb{E}\left[e^{-tY_i^{(n)}}\right] \leq 1 + t^2 p_{i,n} \quad \text{for all } t \in (0, \delta_1).$$

Now, make δ_1 smaller (if necessary) so that $t^2 \leq \frac{t}{4}$. Then, for $t \in (0, \delta_1)$, we have

$$\mathbb{P}\left\{S_n \leq \frac{m_n}{2}\right\} \leq \frac{\mathbb{E}\left[e^{-tR_n}\right]}{e^{\frac{tm_n}{2}}} \quad \text{(Markov Inequality)}$$
$$= e^{\frac{-tm_n}{2}} \prod_{i=1}^{n} \mathbb{E}\left[e^{-tY_i^{(n)}}\right]$$
$$\leq e^{\frac{-tm_n}{2}} \prod_{i=1}^{n}(1 + t^2 p_{i,n})$$
$$= \exp\left(\sum_{i=1}^{n}\left(\frac{-tp_{i,n}}{2} + \log\left(1 + t^2 p_{i,n}\right)\right)\right)$$
$$\leq \exp\left(\sum_{i=1}^{n}\left(\frac{-tp_{i,n}}{2} + t^2 p_{i,n}\right)\right) \quad \text{(since } \log(x) \leq x - 1)$$
$$\leq \exp\left(\frac{-t}{4}\sum_{i=1}^{n} p_{i,n}\right).$$

Since $\limsup_{n \to \infty} m_n = \infty$, we can find a subsequence of $\{m_n\}_{n=1}^{\infty}$, which we call $\{m_{n(k)}\}_{k=1}^{\infty}$, such that $m_{n(k)} \geq k$. Then, $\mathbb{P}\left\{S_{n(k)} \leq \frac{m_{n(k)}}{2}\right\} \leq \exp\left(\frac{-tk}{4}\right)$. On setting $\mathcal{A}_k = \left\{\omega \in \Omega : S_{n(k)}(\omega) \leq \frac{m_{n(k)}}{2}\right\}$, we have

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\mathcal{A}_k\right) < \infty.$$

By the Borel-Cantelli lemma, $\mathbb{P}\{\mathcal{A}_k \ i.o\} = 0$. Thus, $\lim_{k \to \infty} S_{n(k)} = \infty$ w.p.1. Since almost sure convergence implies convergence in distribution, we have $S_\infty = \infty$ w.p.1,

which is a contradiction. Thus, $\limsup_{n \to \infty} m_n < \infty$. In this case, we can find $C < \infty$ such that $m_n \leq C$ for all n. Now, we will show that $\{S_n\}_{n=1}^{\infty}$ is uniformly integrable.

$$\mathbb{E}\left[S_n^2\right] = \mathbb{E}\left[\sum_{i=1}^{n}\left(X_i^{(n)}\right)^2\right] + \mathbb{E}\left[\sum_{\substack{i,j=1 \\ i \neq j}}^{n} X_i^{(n)} X_j^{(n)}\right]$$
$$\leq \sum_{i=1}^{n} p_{i,n} + \sum_{i,j=1}^{n} p_{i,n} p_{j,n}$$
$$= m_n + m_n^2 \leq C\left(1 + C\right) < \infty.$$

Therefore, $\{S_n\}_{n=1}^{\infty}$ is uniformly integrable. By Skorohod's representation theorem, there exists a probability space $(\Omega', \mathcal{F}', P')$ and random variables S_n' and S_∞' having the same distributions as S_n and S_∞, respectively, such that $S_n' \to S_\infty'$ w.p.1. By uniform integrability, we also have L^1 convergence, i.e.,

$$\lim_{n \to \infty} \mathbb{E}\left[S_n'\right] = \mathbb{E}\left[S_\infty'\right]. \quad (9)$$

Therefore, $\lim_{n \to \infty} m_n$ exists and belongs to $(0, \infty)$. ∎

III. CONCLUSION

In this paper, we have analyzed the asymptotic behavior of multiple-access networks containing large numbers of selfish transmitters that share a common wireless communication channel to communicate with their intended receivers. In particular, we have focused on the asymptotic distribution of the total number of packet arrivals to the common wireless channel coming from these selfish transmitters. When selfish transmitters are identical to one another in their utility functions, we have shown that the asymptotic distribution of the total number of packet arrivals becomes equal to a Poisson distribution. On the other hand, when selfish transmitters do not have identical utility functions, we have first obtained a necessary and sufficient condition for the total number of packet arrivals to converge in distribution. We then have shown that the asymptotic packet arrival distribution can be arbitrarily closely approximated in distribution by a summation of finitely many independent Bernoulli random variables and an independent Poisson random variable.

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